INDIAN STATISTICAL INSTITUTE, BANGALORE CENTRE
B.MATH - Second Year, 2023-24, Introduction to Linear Models
Mid-semesteral Examination, February 23, 2024
Marks are shown in square brackets. **Total Marks: 35**
Time: $2\frac{1}{2}$ Hours

1. Suppose $\Sigma = \text{ Cov }(\mathbf{X}) = \begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$ for some random vector X.
(a) Give an example of a random vector $\mathbf{X}$ where Σ has $\rho = 1$. Is it possible to have $\rho = -1$ in Σ?
(b) Show that $-1/3 \leq \rho \leq 1$. [2+5]

2. Suppose $\mathbf{X} \sim N_p(\mathbf{0}, \Sigma)$ where $\text{Rank}(\Sigma) = r \leq p$ and let B and D be symmetric matrices. Show that $\mathbf{X}'B\mathbf{X}$ and $\mathbf{X}'D\mathbf{X}$ are independent χ^2 random variables if and only if
$\Sigma B \Sigma B \Sigma = \Sigma B \Sigma,\ \Sigma B \Sigma D \Sigma = \mathbf{0}$. [7+5]

3. Consider the Gauss-Markov model: $\mathbf{Y} = X\beta + \epsilon$, $E(\epsilon) = 0$, $Cov(\epsilon) = \sigma^2 I_n$. Prove that $\mathbf{a}'\beta$ is estimable if and only if $\mathbf{a}'(X'X)^{-}X'X = \mathbf{a}'$. [5]

4. Consider the following model:
$y_1 = \alpha + \phi + \gamma + \epsilon_1$
$y_2 = \alpha + \phi - \gamma + \epsilon_2$
$y_3 = 2\alpha + 2\phi + \gamma + \epsilon_3$
$y_4 = 2\alpha + 2\phi - \gamma + \epsilon_4$
where α, ϕ, γ are unknown regression parameters and ϵ_i are uncorrelated random variables having mean 0 and variance σ^2.
(a) Does BLUE of $\alpha + \phi - 2\gamma$ exist? Justify. Find it if it exists.
(b) Find the degrees of freedom of RSS. [9+2]